The relations between field coefficients and eigenvector components are

$$\xi_{1n} = \frac{\sum_{r=1}^{L} C_r P'_{nr}}{\cos(k_{x1n}a_1)\frac{1}{2}(a_3 + a'_3)} \qquad (44)$$

$$\xi_{2m} = \frac{-C_m}{\sin(k_{x2m}(a_2 - a_1))}. \qquad (45)$$

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Iterative Solutions of Waveguide Discontinuity Problems

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Abstract—**The method of overlapping regions, together with Schwarz's technique, is applied to waveguide discontinuity problems to illustrate its potential and basic advantages and disadvantages over other methods. The method reasonably corrects an arbitrary initial assumption of field distribution in the plane of discontinuity to the final value in a small number of iterations. The advantages are illustrated for a waveguide bend and dumbbell shaped waveguide as examples of transverse and longitudinal discontinuities, respectively. Numerical results for the case where only the electric field is parallel to the sharp edge discontinuity are presented and compared with available data, while extension to the case where only the magnetic field is parallel to the edge is discussed.**

I. Introduction

SHARP waveguide discontinuities are extensively used in numerous microwave power and communication circuits, and their effects have been under investigation in the last few decades. Generally, these discontinuities are characterized as either transverse or longitudinal, depending on whether the discontinuity lies in a plane transverse or parallel to the direction of propagation, respectively, or both. Waveguide junctions and bends are typical examples of transverse discontinuities, while waveguide complex cross sections belong to the class of longitudinal discontinuities.

Earlier attempts to characterize such discontinuities include rigorous, quasi-rigorous, numerical, and experimental techniques [1]–[4]. The results normally permit computation of scattering matrix parameters, which may be used to evaluate the parameters of an equivalent circuit, cutoff wave numbers, and mode coefficients leading to propagation coefficients and field configurations.

While no method can be expected to deal with the most general case of mixed types of discontinuities and arbitrary waveguide boundaries, the choice of one method over others for the most common discontinuities depends on the shape as well as the electrical and physical dimensions of the waveguide. Thus due to its asymptotic nature, the geometrical theory of diffraction, in which the discontinuity is viewed as multiple body interaction, becomes more appropriate as the smallest linear dimension exceeds one wavelength [5]. However, when the distances between edges and corners are

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smaller than the wavelength, the ray diagram becomes more involved and the accuracy becomes questionable in spite of recent efforts to derive the near field diffraction coefficient [6]. On the other hand, the boundary value approach is only applicable when modal expansions can be found on both sides of the discontinuity plane and does not normally lead to a rapidly convergent series solution when the scattering centers of the discontinuity are electrically close to each other [7]. Other techniques, such as the perturbational method, are limited to small departures in a parameter from the case for which the solution is known. Also, the variational method requires a stationary formula which is relatively insensitive to assumed variation of the field about the correct solution [8]. However, the fact that an expression is stationary does not justify the assumption that it will yield the best approximation when the assumed distribution is completely arbitrary.

Integral techniques, such as those based on the Lorentz reciprocity theorem and the reaction concept, have also been employed but their utility is limited to elementary geometries for which the resulting series or integral equations can be solved [9], [10].

With the availability of fast electronic computers, numerical techniques have been popular and in particular those based on integral or series formulations. This is not only because of their advantages over the differential approach, especially as far as the accuracy and the matrix order are concerned, but also because of their adequacy and conveniency to treat certain scattering and antenna problems [11]. For example, the point matching technique [12], [13] has proven so far to be the most efficient method from the computational time point of view [4]. This technique, however, can only be applied with confidence if the waveguide boundary is such that the Rayleigh hypothesis is valid. Therefore, it is the lack of any systematic method for estimating *a priori* the validity of such hypothesis that appears to be the chief difficulty in applying this technique [13], [14]. Also, like other numerical techniques, it rarely offers more than little physical insight. Therefore, even with the employment of the method of overlapping domains [15], further analytical effort was found necessary to find such relations as that between the eigenvalues of the complementary domains interior and exterior to a regular polygonal conducting cylinder [16].

Since the basic difficulty in waveguide discontinuity problems is to obtain the electric field distribution in the plane of the discontinuity, it is obvious that a method which allows an initial assumption to be iterated in a few steps to the final solution will be attractive. It is, therefore, the purpose of this article to illustrate several advantages of using the method of overlapping regions (OR method) which makes use of such iterations. The OR method generally involves dividing the geometry of the discontinuity into several overlapping subregions for each of which Green's function is known. The solution is then obtained iteratively using Schwarz's iterative procedure [17]. An example for each of transverse and longitudinal discontinuities is given to illustrate the potential and the advantages of the method.

II. Theory

As an example of a transverse discontinuity, we consider the geometry of Fig. 1 where a TE_{10} mode is incident on an assymetrical angled junction (sharp bend) of two H-plane parallel plate waveguides. The geometry is hence two dimensional and may be divided into two subregions I and II of width b_1 and b_2 bounded by S_1 and S_2, respectively, and which overlap in the shaded area A.

Applying Green's theorem in region I, it is clear that the total electric field E_{zI} in terms of the coordinates (x_1,y_1) is given by

$$E_{zI}(x_1,y_1) = E_z^i(x_1,y_1) + \int_{S_1} \{G_I(n \cdot \nabla E_{zI}^s) - E_{zI}^s(n \cdot \nabla G_I)\}\, ds \quad (1)$$

where E_{zI}^s is the scattered field satisfying the two-dimensional wave equation, G_I is Green's function for an infinite parallel plate region. E_z^i is the incident electric field of unit amplitude and is given by

$$E_z^i(x_1,y_1) = \sin\left(\frac{\pi y_1}{b_1}\right) \exp\,(jk_1 x_1) \quad (2)$$

where $k_n = [k^2 - (n\pi/b_1)^2]^{1/2}$ and the $\exp\,(-j\omega t)$ time dependence has been omitted. If G_I is chosen to satisfy the Dirichlet boundary condition in region I, namely,

$$G_I(x_1,y_1/x_{10},y_{10}) = \frac{-j}{b_1} \sum_{n=1}^{\infty} \frac{1}{k_n} \exp\,(-jk_n|x_1 - x_{10}|) \cdot \sin\left(\frac{n\pi y_1}{b_1}\right) \sin\left(\frac{n\pi y_{10}}{b_1}\right) \quad (3)$$

where (x_{10},y_{10}) are the coordinates of the source point, then the integration limits in (1) will only involve the boundaries l_1 and l_2 of A as shown in Fig. 1. By combining (1)–(3) and substituting $E_{zI} - E_z^i$ for E_{zI}^s, the total electric field in region I may be expressed in terms of its value and its normal derivative on l_1 and l_2, respectively,

$$\begin{aligned} E_I(x_1,y_1) = E^i &+ \frac{j\pi}{b_1^2} \sum_{n=1}^{\infty} \frac{n(-1)^n}{k_n} \sin\left(\frac{n\pi y_1}{b_1}\right) \cdot \frac{\int_{-b_2+b_1\cos(\pi-\alpha)}^{b_1\cot(\pi-\alpha)} E_I\Big|_{y_{10}=b_1}}{\sin\,(\pi-\alpha)} \cdot \exp\,(-jk_n|x_1 - x_{10}|\; dx_{10}) \\ &- \frac{j}{b_1} \sum_{n=1}^{\infty} \frac{1}{k_n} \sin\left(\frac{n\pi y_1}{b_1}\right) \int_{b_1\cot(\pi-\alpha)}^{0} \left(\frac{\partial E_I}{\partial n_1} - \frac{\partial E^i}{\partial n_1}\right) \cdot \sin\left(\frac{n\pi y_{10}}{b_1}\right)\Bigg|_{y_{10}=x_{10}\tan(\pi-\alpha)} \cdot \exp\,(-jk_n|x_1 - x_{10}|\; dx_{10}) \\ &+ \int_{b_1\cot(\pi-\alpha)}^{0} E^i n_1 \cdot \nabla G_I\Big|_{y_{10}=x_{10}\tan(\pi-\alpha)} dx_{10} \end{aligned} \quad (4)$$

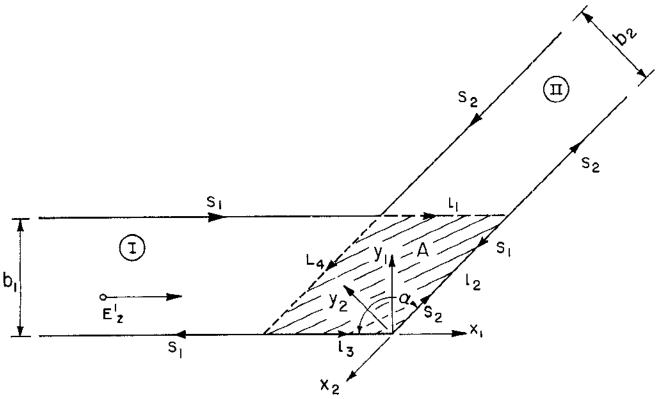

Fig. 1. Longitudinal section of a sharp bend between two assymetrical H-plane waveguides.

where the z subscript has been dropped and E^i is given in (2). The outward unit vector n_1 is normal to l_2 and is given by

$$n_1 = \sin(\pi - \alpha)\hat{x}_1 - \cos(\pi - \alpha)\hat{y}_1 \qquad (5)$$

where $\hat{x}_1$ and $\hat{y}_1$ are unit vectors along the x_1 and y_1 axes, respectively.

Similarly, the total field in region II is given in terms of the (x_2,y_2) coordinates by

$$E_{\rm II}(x_2,y_2) = \int_{S_2} \{G_{\rm II}(n \cdot \nabla E_{\rm II}) - E_{\rm II}(n \cdot \nabla G_{\rm II})\}\, ds. \qquad (6)$$

The boundary S_2 will only involve l_3 and l_4 of A if $G_{\rm II}$ is chosen as before to satisfy the Dirichlet boundary condition, then b_1 in (3) is replaced by b_2 and k_n by

$$k_n' = [k^2 - (n\pi/b_2)^2]^{1/2}. \qquad (7)$$

Schwarz's iterative method of solution is initiated by assuming the electric field on l_4 and its normal derivative on l_3 which are denoted by $E_{\rm II}(l_4)$ and $E_{\rm II}'(l_3)$, respectively. Although an arbitrary assumption is possible, the incident field is suggested to reduce the number of iterations as will be shown in the numerical results. Hence, the assumed values on l_3 and l_4 as calculated from (2) can be used to calculate from (6) the field on l_1 and its normal derivative on l_2, i.e., $E_{\rm I}(l_1)$ and $E_{\rm I}'(l_2)$, which are then substituted back in (4). A second-order approximation for the initially assumed field can next be calculated. Although the iterative procedure continues in a similar manner, it should be noted that after each iteration the calculated $E_{\rm I}(l_1)$ and $E_{\rm I}'(l_2)$ should be normalized [18] so as to maintain an incident field of constant amplitude.

Another example to illustrate a longitudinal-type discontinuity is the dumbbell waveguide cross section shown in Fig. 2. Due to the nature of the geometry an integral equation formulation in terms of Green's function rather than a series eigenfunction expansion in each of the subregions is preferable [19]–[21]. Since $x = 0$ represents either an electric or magnetic symmetry plane, we consider TM modes where the electric field is z-polarized everywhere. For the case of an electric symmetry plane at $x = 0$, the cross section on either side is divided into two subregions. These consist of region I (with the coordinate system at origin 0_1) which is a finite rectangle of dimensions b and d, and region II (with the coordinate system at origin 0_2) which is a circle of radius a. These two subregions overlap in the shaded area as shown in Fig. 2.

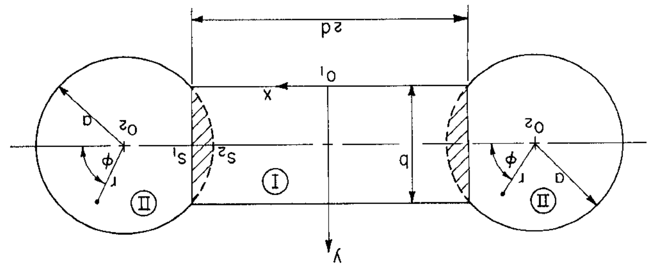

Fig. 2. Cross section of the dumbbell waveguide.

Applying Green's theorem in region I, the electric field for TM modes is given by

$$E_{\rm I}(x,y) = \int_{S_1} E_{\rm I}(d,y_0)\frac{\partial G_{\rm I}^{(e)}}{\partial x_0}\, dy_0. \qquad (8)$$

Here $G_{\rm I}^{(e)}$ is Green's function satisfying the Dirichlet boundary condition in region I and is therefore given by

$$G_{\rm I<}^{(e)}(x,y/x_0,y_0) = \frac{2}{d}\sum_{m=1}^{\infty} \sin(m\pi x/d)\sin(m\pi x_0/d) \cdot \sin(\gamma_m y)\sin(\gamma_m(b - y_0))/\gamma_m \sin(\gamma_m b) \qquad (9)$$

where $\gamma_m = [k_c^2 - (m\pi/d)^2]^{1/2}$ and (x_0,y_0) are the coordinates of a singular source distribution due to the nonzero field on the boundary S_1. Similarly, the electric field at any point interior to region II is given by

$$E_{\rm II}(r,\phi) = -\int_{S_2} E_{\rm II}(a,\phi_0)\frac{\partial G_{\rm II}^{(e)}}{\partial r_0}\, a\, d\phi_0. \qquad (10)$$

Here $G_{\rm II}^{(e)}$ is Green's function satisfying the Dirichlet boundary condition in region II and is therefore given by

$$G_{\rm II<}^{(e)}(r,\phi/r_0,\phi_0) = \frac{j}{4}\sum_{m=-\infty}^{\infty} \exp(jm(\phi - \phi_0)) \cdot \left[H_m^{(1)}(k_c r_0) - \frac{H_m^{(1)}(k_c a)}{J_m(k_c a)} J_m(k_c r_0)\right] J_m(k_c r). \qquad (11)$$

To determine the cutoff wavenumber k_c the arguments of $E_{\rm I}$ are first changed to (r,ϕ) and the continuity condition on the electric field and its normal derivative at S_2 are then applied to eliminate the integral term in (10) using the orthogonality of the trigonometric functions. It should be noted that in doing so both $G_{\rm I\gtrless}^{(e)}$ in (9) are to be used in (8). This is simply because for each point P on S_2 we calculate y_p and x_p and hence $G_{\rm I<}^{(e)}$ is used for $y_0 > y_p$ while $G_{\rm I>}^{(e)}$ is used for $y_0 < y_p$. This results in expressing the integration limits in (8) in two parts from 0 to y_p and from y_p to b. This leads to an integral equation in which $E_{\rm I}(d,y_0)$ and k_c are the unknowns and may be determined iteratively. This is achieved by first assuming a value of $E_{\rm I}(d,y_0)$ in order to compute an approximate k_c so that $E_{\rm II}(a,\phi_0)$ may be calculated from (8). The initial results are then improved by substituting $E_{\rm II}(a,\phi_0)$ in

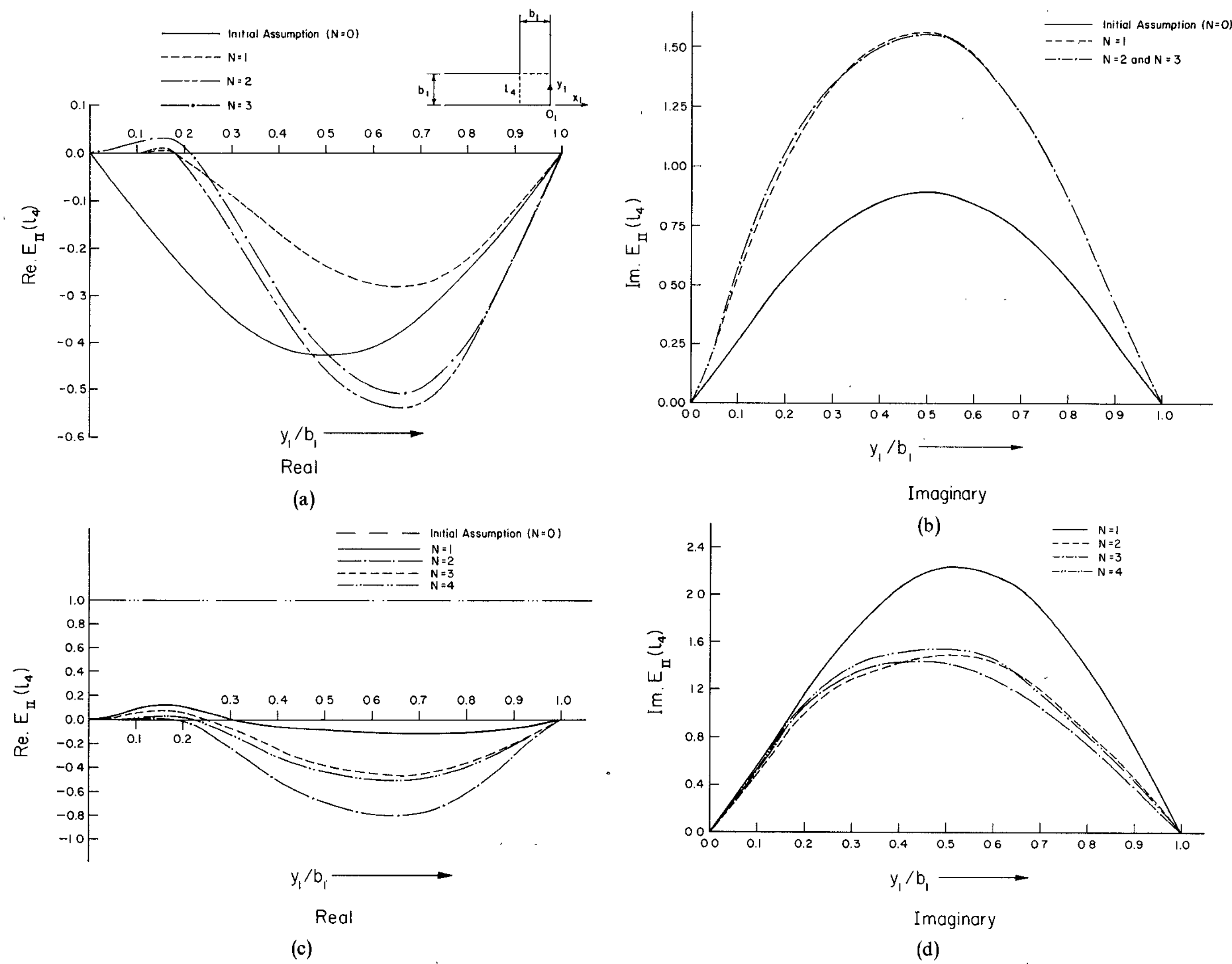

Fig. 3. Convergence of the field distribution at the boundary l_4 of a symmetrical right-angled H-plane corner for two different initial assumptions. (a) and (b) for an incident field of unity maximum amplitude. (c) and (d) for a constant field of unity amplitude.

(10) to calculate a second-order approximation of $E_I(d,y_0)$.

Finally, the extension to the TM modes with a magnetic symmetry plane are straightforward and need not be repeated.

III. Numerical Results

In order to confirm the accuracy of the method, we present numerical results in Table I for the reflection coefficient of a right-angled corner together with experimental data. The computed results on the basis of (4) are found to be unaffected for this special case if the Green's function for a semi-infinite (rather than infinite) parallel plate waveguide is used. As a result, the computation time is considerably reduced and amounted to 3.5 s for each value of kb_1 using IBM 370/158 electronic computer. For this the series in (3) was truncated to five terms and the 3-point Newton–Cotes formula of Simpson's rule was employed for integration with an interval of 0.05λ between the successive points. The number of iterations N naturally depends on the initial choice of the electric field as illustrated in Table II for two different initial assumptions while Fig. 3 illustrates the corresponding convergence of the field on l_4. Table III shows a comparison for the cutoff wavenumber in the special case of a cigar-shaped waveguide with previously published data. This is the only case available for comparison in order to confirm the accuracy of the method in the dumbbell waveguide example and is obtained by letting $b = 2a$ in Fig. 2. The required computation time per iteration for the cutoff wavenumber k_c of the dominant mode is 3.6 s using the same computer and when the series in (9) is truncated after the third term ($m = 3$). The d/a values are kept larger than one to avoid overlapping of the two circles in Fig. 2.

IV. Discussion

Examination of our results shows that the method converges regardless of the initial assumption of the unknown field on the boundaries of the overlapping area. However, the number of iterations is reduced if the initial assumption is reasonable, as could be obtained from the incident field. The fact that an arbitrary initial assumption is allowed is

TABLE I
REFLECTION COEFFICIENT OF THE TE_{10} MODE INCIDENT ON A SYMMETRICAL RIGHT-ANGLED H-PLANE CORNER

kb_1	Computed	Measured (Magnitude)
5.0	0.499 /−150.6°	0.493
5.1	0.539 /−153.7°	0.533
5.2	0.584 /−156.6°	0.560
5.3	0.643 /−158.5°	0.616
5.4	0.705 /−159.0°	0.697

TABLE II
CONVERGENCE OF THE REFLECTION COEFFICIENT FOR TWO DIFFERENT INITIAL ASSUMPTIONS ($kb_1 = 5.3$)

	Initial Assumption for $E_{II}(l_4)$	
N	Incident Field of Unity Maximum Amplitude	Constant Field of Unity Amplitude
1	$-0.699 - j0.111$	$-1.359 - j0.222$
2	$-0.611 - j0.294$	$-0.471 - j0.471$
3	$-0.571 - j0.230$	$-0.529 - j0.146$
4	$-0.602 - j0.225$	$-0.639 - j0.220$
5	$-0.601 - j0.237$	$-0.597 - j0.251$

TABLE III
CUTOFF WAVENUMBERS $k_c a$ OF THE DOMINANT TM MODE IN A CIGAR-SHAPED WAVEGUIDE

d/a	Finite Elements Method from Graphical Data [22]	OR Method $m = 1$, $N = 1$	$m = 1$, $N = 3$	$m = 3$, $N = 3$
1.15	1.78	2.040	2.014	1.862
1.17	1.77	1.985	1.943	1.817
1.20	1.76	1.965	1.930	1.803
1.23	1.755	1.861	1.800	1.762
1.25	1.75	1.845	1.793	1.756
1.3	—	1.739	1.742	1.741
1.5	—	1.335	1.387	1.483

simply because each iteration replaces the values of the previous iteration by a better estimate rather than adding a correction term as normally done in the geometric theory of diffraction. As a result, a sinusoidal field distribution for the TE_{10} mode incident on the corner and which is the incident field resulted in a correct field distribution after five iterations. In this sense the method can be used to correct the incident field used in the Kirchhoff theory of diffraction as the aperture field can be replaced by a more accurate value resulting from the iterations. In another sense the method can help in finding numerical values for the diffraction coefficients, used in the geometrical theory of diffraction, of closely separated edges [23]. These are usually unavailable due to the lack of knowledge of the wavefront in near field interaction.

It should be noted that although we have placed confidence in computing the cutoff wavenumbers of the fundamental TM mode, the extension to higher order modes is straightforward as reported elsewhere [20], [21]. Also, the extension of the formulation to TM modes in the case of Fig. 1 (e.g., by changing the excitation) and TE modes in the case of Fig. 2 can be easily achieved in terms of the axial magnetic field and following a similar procedure and employing the Green's functions for the Neumann boundary condition. Numerical difficulties which are encountered because of the singular behavior of the magnetic field near the edges depending on the angle of the corner can, however, be dealt with by decomposing the magnetic field near the edges into two components. The first component describes the order of the singularity in a manner similar to that previously employed [24], [25], while the second is straightforward since it describes the regular behavior of the field. Analytical integration of the former component will eliminate the numerical difficulties in a manner similar to that followed by Harrington [26].

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Analysis of a Waveguide Hybrid Junction by Rank Reduction

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Abstract—**Exact equations characterizing a waveguide hybrid junction traversed by a dielectric sheet are formulated by waveguide field-equivalence decomposition. A new reduced-rank spectral expansion technique avoids inversion of a large ill-conditioned matrix in the calculation of the scattering matrix. Arbitrary sheet thicknesses and permittivities are treated, accounting fully for waveguide boundaries and offset. For illustrative purposes, numerical results are presented for a rectangular waveguide hybrid, when only the dominant mode propagates.**

I. Introduction

THE INTEGRAL EQUATIONS, and the corresponding matrix equations, that represent scattering at a waveguide discontinuity often exhibit ill-conditioned behavior. In a previous paper [1] it was shown that the resultant difficulties can be largely overcome by taking advantage of the relatively low effective rank of the ill-conditioned portion of the matrix. In the following sections the new rank-reduction technique is applied to the waveguide hybrid junction problem. The hybrid junction, an important component of certain advanced microwave communication systems [2], [3] has resisted accurate analysis.

The hybrid junction to be analyzed is shown in Fig. 1. It consists of two crossed waveguides whose junction is traversed by a dielectric sheet at a 45° angle. By properly choosing the dielectric constant and sheet thickness, a directional coupler can be formed for a given frequency band. This coupler is used in band diplexing networks [4], [5]. An accurate analysis of the hybrid is important in the design of the band diplexer to achieve satisfactory frequency band separation and to identify spurious modes that may degrade system performance.

This paper will serve several purposes. It will illustrate the use of field superposition principles to decompose the complicated geometry of a hybrid junction into a combination of separate, uniform waveguides. It will discuss the application of this technique to derive an exact set of equations for the scattering matrix of the junction. It will further illustrate that, for those frequency intervals in which higher order modes affect the scattering significantly, their effect can be calculated despite the ill-conditioning of the equations, by use of a new rank-reduction technique. Finally, it will present the scattering coefficients for the hybrid, in a frequency range in which quasi-optical approximations are not valid.

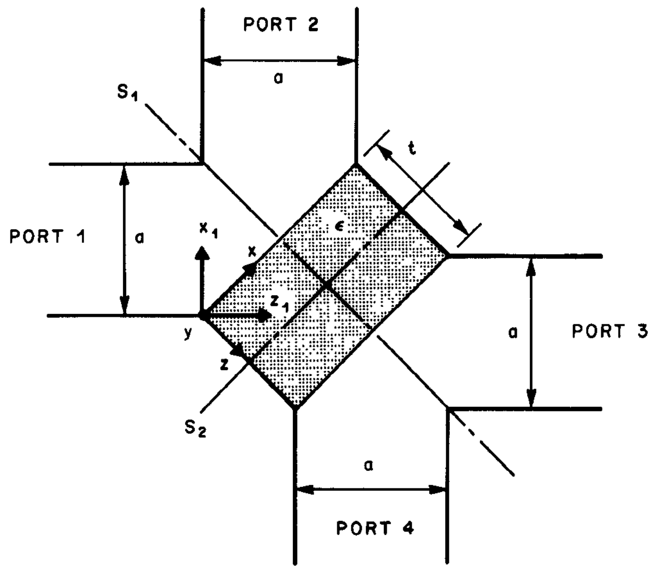

Fig. 1. Top view of a hybrid junction formed by two crossed waveguides of width a whose junction is traversed at a 45° angle by a dielectric sheet of thickness t and relative dielectric constant ε.

The analysis presented here is not restricted by the waveguide geometry, and permits the computation of higher order mode coupling. For purposes of illustration of the new technique, numerical results are presented for a rectangular waveguide hybrid over a frequency range in which only the

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